Black-Scholes Option Pricing Formula:

Stock Process:

$$S(t) = S(0)\ \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}, \quad t \geq 0 \tag{1}$$

European Option's Non-Arbitrage Price:

$$G(S(t),\, t) = \mathbf{E}^{\mathbf{Q}}[e^{-r(T-t)}G(S(T),\, T) \mid S(t)] \tag{2}$$

European Call Option:

$$G(S(T),\, T) = (S(T) - K)^+ \tag{3}$$

$$\begin{aligned} G(S(t),\, t) &= \mathbf{E}^{\mathbf{Q}}[e^{-r(T-t)}(S(T)-K)^+ \mid S(t)] \\ &= e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\left[\left(S(t)\ \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma W(T-t)\right\} - K\right)^+ \mid S(t)\right] \end{aligned} \tag{4}$$

$W(T-t) \perp S(t)$, and $\{W(t),\, t \geq 0\}$ is Brownian Motion under measure $\mathbf{Q}$, which means

$$W(T-t) \sim \mathcal{N}(0,\, T-t) \tag{5}$$

$$(4) = e^{-r(T-t)}\int_{z_0}^{\infty}\left(S(t)\ \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\cdot z\right\} - K\right)\varphi(z)\,dz \tag{6}$$

Where $\varphi(z) = \frac{1}{\sqrt{2\pi}}\ \exp\left\{-\frac{z^2}{2}\right\}$

$$S(t)\ \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\cdot z_0\right\} = K \tag{7}$$

$$z_0 = \frac{\log\left(\frac{K}{S(t)}\right) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{8}$$

$$(6) = \underbrace{e^{-r(T-t)}\int_{z_0}^{\infty} S(t)\ \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\sqrt{T-t}\cdot z\right\}\varphi(z)\,dz}_{\text{Part I}} - \underbrace{e^{-r(T-t)}\int_{z_0}^{\infty} K\varphi(z)\,dz}_{\text{Part II}} \tag{9}$$

Part I:

$$\begin{aligned} \text{Part I} &= S(t)\int_{z_0}^{\infty}\exp\left\{-\frac{1}{2}\sigma^2(T-t) + \sigma\sqrt{T-t}\cdot z\right\}\frac{1}{\sqrt{2\pi}}\ \exp\left\{-\frac{z^2}{2}\right\}dz \\ &= S(t)\int_{z_0}^{\infty}\frac{1}{\sqrt{2\pi}}\ \exp\left\{-\frac{z^2 - 2\sigma\sqrt{T-t}\cdot z + \sigma^2(T-t)}{2}\right\}dz \\ &= S(t)\int_{z_0}^{\infty}\frac{1}{\sqrt{2\pi}}\ \exp\left\{-\frac{(z - \sigma\sqrt{T-t})^2}{2}\right\}dz \\ &= S(t)\cdot\mathbf{P}[\,Z' \geq z_0\,] \end{aligned} \tag{10}$$

Where $Z' \sim \mathcal{N}(\sigma\sqrt{T-t},\, 1)$,

$$(10) = S(t)\cdot\mathbf{P}[\,Z'' \geq (z_0 - \sigma\sqrt{T-t}\,)\,] \tag{11}$$

Where $Z'' = Z' - \sigma\sqrt{T-t} \sim \mathcal{N}(0,\,1)$,

$$\begin{aligned}
\text{Part I} &= S(t)\cdot\mathbf{P}\left[\,Z'' \,\geq\, \frac{\log\left(\frac{K}{S(t)}\right) - \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\,\right] \\
&= S(t)\cdot\mathcal{N}(d_1) \qquad (12)
\end{aligned}$$

Where:

$$\begin{aligned}
d_1 &= -\left(\frac{\log\left(\frac{K}{S(t)}\right) - \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \frac{\log\left(\frac{S(t)}{K}\right) + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad (13)
\end{aligned}$$

Part II:

$$\text{Part II} = K\,e^{-r(T-t)}\cdot\mathbf{P}\left[\,Z \,\geq\, z_0\,\right] \qquad (14)$$

Where $Z \sim \mathcal{N}(0,\,1)$

$$\text{Part II} = K\,e^{-r(T-t)}\cdot\mathcal{N}(d_2) \qquad (15)$$

Where:

$$\begin{aligned}
d_2 &= -z_0 \\
&= \frac{\log\left(\frac{S(t)}{K}\right) + \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad (16) \\
&= d_1 - \sigma\sqrt{T-t}
\end{aligned}$$

European Call Option:

$$C(S(t),\,t) = S(t)\cdot\mathcal{N}(d_1) - K\,e^{-r(T-t)}\cdot\mathcal{N}(d_2) \qquad (17)$$

$$d_1 = \frac{\log\left(\frac{S(t)}{K}\right) + \left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad (18)$$

$$d_2 = \frac{\log\left(\frac{S(t)}{K}\right) + \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \qquad (19)$$

European Put Option:
Put-Call Parity:

$$C(S(t),\,t) + K\,e^{-r(T-t)} = P(S(t),\,t) + S(t) \qquad (20)$$

$$\begin{aligned}
P(S(t),\,t) &= S(t)\cdot\mathcal{N}(d_1) - K\,e^{-r(T-t)}\cdot\mathcal{N}(d_2) + K\,e^{-r(T-t)} - S(t) \\
&= S(t)\cdot(\mathcal{N}(d_1)-1) + K\,e^{-r(T-t)}\cdot(1-\mathcal{N}(d_2)) \\
&= -S(t)\cdot(1-\mathcal{N}(d_1)) + K\,e^{-r(T-t)}\cdot\mathcal{N}(-d_2) \\
&= -S(t)\cdot\mathcal{N}(-d_1) + K\,e^{-r(T-t)}\cdot\mathcal{N}(-d_2) \\
&= K\,e^{-r(T-t)}\cdot\mathcal{N}(-d_2) - S(t)\cdot\mathcal{N}(-d_1) \qquad (21)
\end{aligned}$$

European Put Option:

$$K\,e^{-r(T-t)}\cdot\mathcal{N}(-d_2) - S(t)\cdot\mathcal{N}(-d_1) \qquad (22)$$

$$d_1 = \frac{\log\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{23}$$

$$d_2 = \frac{\log\left(\frac{S(t)}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{24}$$